Witten index $\mathrm{tr}\,(-1)^F e^{-\beta H} = n^B_{E=0} - n^F_{E=0}$

How it captures the geometry of field config. space?

example 1.

Hilbert space $\ni \psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$

$$(\psi, \psi) = \int_{-\infty}^{\infty} dx \, (|\phi_1|^2 + |\phi_2|^2)$$

$\begin{pmatrix} 0 \\ \phi_2(x) \end{pmatrix}$: Boson , $\begin{pmatrix} \phi_1(x) \\ 0 \end{pmatrix}$: Fermion.

$$(-1)^F = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = -\sigma_3 \quad : \text{Pauli matrix}$$

Supercharge

$$Q = \frac{1}{2}(\sigma_1 p + \sigma_2 V'(x))$$

$$p = -i\hbar \frac{d}{dx}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$Q^\dagger = Q$$

$$Q = \frac{1}{2}(\sigma_+ (p + iV') + \sigma_- (p - iV'))$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

<u>Boson $\rightarrow$ Fermion</u> <u>Fermion $\rightarrow$ Boson</u>

$$(\sigma_+)^2 = (\sigma_-)^2 = 0, \qquad \{\sigma_+, \sigma_-\} = 1$$

Hamiltonian

$$H = \{Q, Q\} = \frac{1}{2}\left[p^2 + (V'(x))^2 + \hbar \sigma_3 V''(x) \right]$$

$H \geq 0$ by construction

component-wise

$$\frac{1}{2} p^2 + \frac{1}{2} (V'^2 + \hbar V'') \quad \text{for Fermion}$$

$$\frac{1}{2} p^2 + \frac{1}{2} (V'^2 - \hbar V'') \quad \text{for Boson}$$

Ground states

- perturbation

$\hbar \to 0$; ground state $\Leftrightarrow$ $V'(x) = 0$

Suppose $V'(x) = \lambda (x - x_0) + O((x-x_0)^2)$

Fermion ; $\frac{1}{2} p^2 + \frac{1}{2} \lambda^2 (x - x_0)^2 + \frac{1}{2}\hbar\lambda + \cdots$

Boson : $\frac{1}{2} p^2 + \frac{1}{2} \lambda^2 (x - x_0)^2 - \frac{1}{2}\hbar\lambda + \cdots$

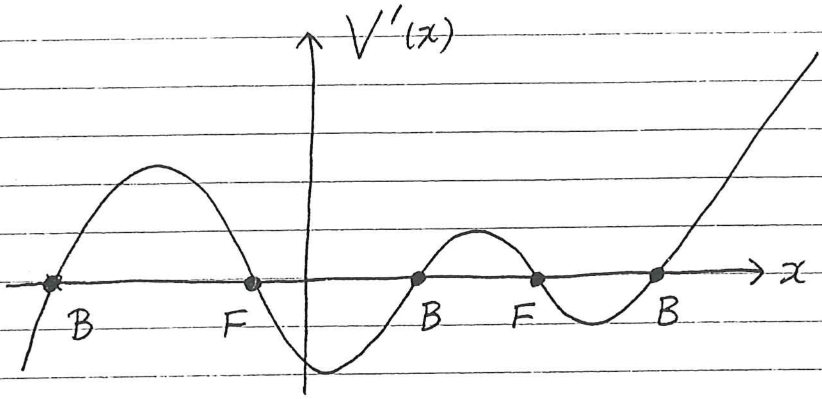
To the leading order in $\hbar$

ground state energy for Fermion $\cdots \frac{1}{2}\hbar(|\lambda| + \lambda)$

Boson $\cdots \frac{1}{2}\hbar(|\lambda| - \lambda)$

Therefore $\lambda > 0 \Rightarrow$ zero-energy state = Bosonic

$\lambda < 0$ " = Fermionic



i.e. $$n^{B}_{E=0} = \#\,(x : V'(x) = 0,\ V''(x) > 0)$$
$$n^{F}_{E=0} = \#\,(x : V'(x) = 0,\ V''(x) < 0)$$

This result is true to all order in the perturbation.

... non-renormalization theorem

Is this true even non-perturbatively?

— No!

We can check this by solving
$H\psi(x) = 0$ explicitly.

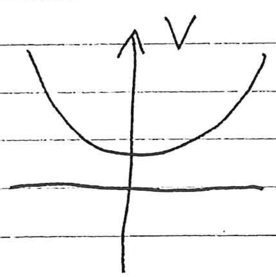
$$H\psi = 0 \Leftrightarrow Q\psi = \frac{1}{2}\left(\sigma_+ (p+iV') + \sigma_- (p-iV')\right)\psi$$

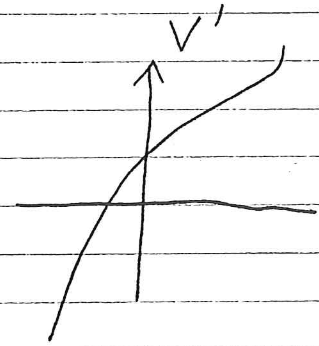
$$= \frac{\hbar}{2i}\begin{pmatrix} \left(\frac{d}{dx} - V'(x)\right)\phi_2 \\ \left(\frac{d}{dx} + V'(x)\right)\phi_1 \end{pmatrix} = 0$$

$$\begin{cases} \phi_1 = \exp(-V(x)+V(0))\,\phi_1(0) \\ \phi_2 = \exp(+V(x)-V(0))\,\phi_2(0) \end{cases}$$

The solution to $H\psi = 0$ is unique, if exists.

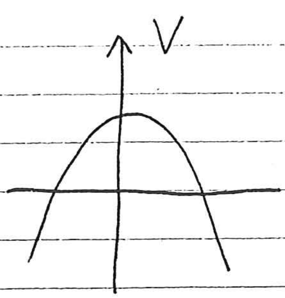
<u>Normalizability test</u>

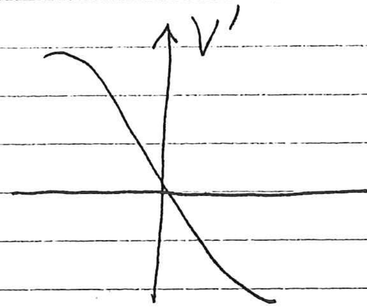
(1) 

i.e. 

$\Rightarrow \psi = \begin{pmatrix} e^{-V(x)} \\ 0 \end{pmatrix}$

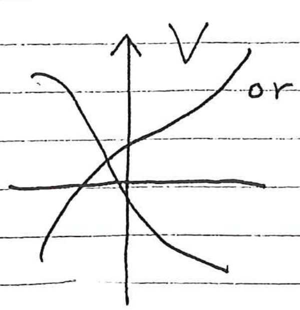
Fermionic

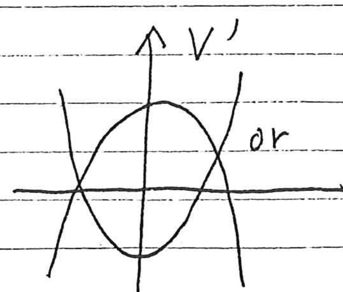
(2) 

i.e. 

$\Rightarrow \psi = \begin{pmatrix} 0 \\ e^{V(x)} \end{pmatrix}$

Bosonic

(3) 

i.e. 

$\Rightarrow$ <u>No Solution</u>

Lesson:

- The perturbative computation of $n^B_{E=0}$ and $n^F_{E=0}$ cannot be trusted,

  modified by the instanton correction.

- The computation of $\mathrm{tr}\,(-1)^F e^{-\beta H} = n^B_{E=0} - n^F_{E=0}$ is exact.

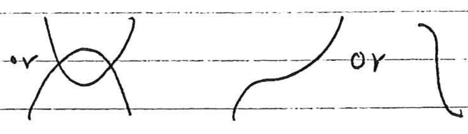
N.B.

$$\#(x : \lambda > 0) - \#(x : \lambda < 0)$$

$$= 0,\ \pm 1$$

Topological inv

local expression.



xample 2. sigma-model in two dimensions

M : Riemannian mfd.

$$S = \frac{1}{2}\int d^2z\, d^2\theta\; g_{ij}(\Phi)\, D\Phi^i \bar{D}\Phi^j$$

$$= \int d^2z \Big(\frac{1}{2} g_{ij}\, \partial\varphi^i \bar{\partial}\varphi^j + \frac{i}{2} g_{ij}\, \psi^i (\bar{\partial}\psi^j + \Gamma^j_{kl}\, \bar{\partial}\varphi^k \psi^l)$$

$$+ \frac{i}{2} g_{ij}\, \bar{\psi}^i (\partial\bar{\psi}^j + \Gamma^j_{kl}\, \partial\varphi^k \bar{\psi}^l)$$

$$+ \frac{1}{8} R_{ijkl}\, \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l \Big)$$

$\mathrm{tr}\,(-1)^F e^{-\beta H}$

(A) perturbation

(B) reduction to one-dimension $\Rightarrow$ exact computation (in the spirit of example 1).

(C) reduction to zero dimension

(A) $\Leftrightarrow$ (B) : Morse theory

(B) $\Leftrightarrow$ (C) : index theorem. (Gauss-Bonnet.)

(A) S has continuously degenerate vacua:

$$(\ \Phi(x,t) = \varphi_0 \quad \text{for any } \varphi_0 \in M)$$

$$\Downarrow$$

Lift the degeneracy by adding

$\Delta S = \int d^2z\, d^2\theta\, h(\Phi)$ to the action.

$h(\Phi)$: $\forall$ function on M, supersymmetric perturbations

This does not change $\mathrm{tr}(-1)^F e^{-\beta H}$

as far as $h(\varphi)$ is $\left(\begin{array}{c}\text{finite}\\ \text{regular}\end{array}\right)$ everywhere on M

$$\Delta S = \int d^2z\,(-1)\cdot\left(\frac{1}{2}g^{ij}\frac{\partial h}{\partial\varphi^i}\frac{\partial h}{\partial\varphi^j} + \frac{1}{2}\frac{\partial^2 h}{\partial\varphi^i\partial\varphi^j}\psi^i\bar{\psi}^j\right)$$

$$\left[\ \text{C.F.}\quad H = \frac{1}{2}p^2 + \frac{1}{2}(V'^2 + \sigma_3 V'') \ \text{ in example 1.}\ \right]$$

$$V' \sim g^{ij}\frac{\partial h}{\partial\varphi^i}\frac{\partial h}{\partial\varphi^j}$$

$$V'' \sim \frac{\partial^2 h}{\partial\varphi^i\partial\varphi^j}$$

ground states are at critical points $\frac{\partial h}{\partial \varphi^i} = 0$

$$h(\varphi) = \frac{1}{2} m_{ij} (\varphi^i - \varphi_0^i)(\varphi^j - \varphi_0^j) + \cdots$$

$$|\text{vacuum}\rangle_{\varphi_0} = \underbrace{|B\rangle \otimes \cdots \otimes |B\rangle}_{\text{pos. eigenvalues of } m} \otimes \underbrace{|F\rangle \otimes \cdots \otimes |F\rangle}_{\text{negative eigenvalues of } m}$$

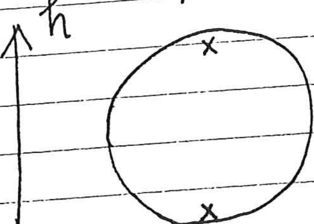
$$(-1)^F |\text{vacuum}\rangle_{\varphi_0} = (-1)^{\overbrace{\#(\text{negative eigenvalues of } m)}^{\text{Morse index}}} |\text{vacuum}\rangle_{\varphi_0}$$

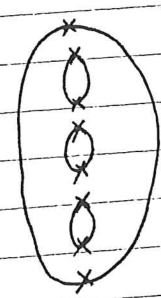
Therefore, to the leading order in $\hbar$,

$$\underbrace{\operatorname{tr}(-1)^F e^{-\beta H} = \sum_{\varphi_0 : \left.\frac{\partial h}{\partial \varphi}\right|_{\varphi_0} = 0} (-1)^{\#\left(\text{n.e.v. of } \left.\frac{\partial^2 h}{\partial \varphi \partial \varphi}\right|_{\varphi_0}\right)}}_{\substack{\text{Does not depend on the choice of } h(\varphi) \\ \text{(Morse theory).}}}$$

e.g.

sphere





$\operatorname{tr}(-1)^F = 1 + 1$
$= 2$

$\operatorname{tr}(-1)^F = 1 - 2g + 1 = 2 - 2g$
$=$ Euler number.

(B) reduction to one-dimensional problem.

$$S = \underbrace{L}_{\substack{\text{space}\\ \text{volume}}} \int dt \Big(\frac{1}{2} g_{ij}(\varphi)\dot{\varphi}^i \dot{\varphi}^j$$

$$+ \frac{i}{2} g_{ij}(\varphi) \bar{\psi}^i \Big(\frac{d}{dt}\psi^j + \Gamma^j_{kl}\dot{\varphi}^k \psi^l \Big)$$

$$+ \frac{1}{4} R_{ijkl}\psi^i \psi^j \bar{\psi}^k \bar{\psi}^l \Big)$$

Canonical Quantization

$$\Rightarrow \quad \{\psi^i, \psi^j\} = 0, \quad \{\bar{\psi}^i, \bar{\psi}^j\} = 0$$

$$\{\psi^i, \bar{\psi}^j\} = g_{ij}(\varphi)$$

$\psi^i|0\rangle = 0 \quad \Rightarrow \quad |0\rangle,\ \bar{\psi}^i|0\rangle,\ \bar{\psi}^i\bar{\psi}^j|0\rangle,\ \cdots,\ \bar{\psi}^{i_1}\cdots\bar{\psi}^{i_n}|0\rangle$

$n = \dim M.$

Hilbert space $\ni A_{i_1\cdots i_m}(\varphi)\bar{\psi}^{i_1}\cdots\bar{\psi}^{i_m}|0\rangle$

$\wr\!\wr \qquad\qquad \Updownarrow$

Space of forms $\quad A_{i_1\cdots i_m} dx^{i_1}\wedge\cdots\wedge dx^{i_m}$: m-form

$|0\rangle \Leftrightarrow 1 \qquad (-1)^F = +1$

$\bar{\psi}^i|0\rangle \Leftrightarrow dx^i \qquad (-1)^F = -1$

$\bar{\psi}^i\bar{\psi}^j|0\rangle \Leftrightarrow dx^i\wedge dx^j \qquad (-1)^F = +1$

Supersymmetry

$$\begin{cases} \delta\varphi^i = i\varepsilon\psi^i \\ \delta\psi^i = -\varepsilon(\dot{\varphi}^i + i\Gamma^i_{jk}\bar{\psi}^j\psi^k) \end{cases}$$

Supercharge $\quad Q = i p_i \bar{\psi}^i, \quad Q^\dagger = -i p_i \psi^i$

$$Q \leftrightarrow dx^i \frac{\partial}{\partial x^i} = d \quad : \text{ exterior derivative}$$

m-form $\to$ $(m+1)$-form

$$Q^\dagger \leftrightarrow d^*$$

$(m+1)$-form $\to$ m-form

$$H = \{Q, Q^\dagger\} \leftrightarrow \Delta = d d^* + d^* d.$$

$\#$ (ground state at degree m) $= b_m$: m^{th} Betti number.

$$\operatorname{tr}(-1)^F e^{-\beta H} = \sum_{m=0}^{n} (-1)^m b_m = \chi(M).$$

Thus we have proved, combining (A) & (B), that

$$\sum_{\varphi_0 : \partial h = 0} (-1)^{\#(n.e.v.\text{ of } \partial^2 h(\varphi_0))} = \chi(M)$$

for any $h(\varphi)$ on M.

$\boxed{\text{N.B.}}$ To leading order in $\hbar$, $b_m = \#(x : \# n.e.v = m)$.
This is modified by instanton correction.

(C) reduction to zero-dimensional problem.

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \frac{1}{\sqrt{(2\pi)^n}} \int d^n x \overbrace{\prod_{m=1}^{n} d\psi^m d\bar{\psi}^m}^{\text{zeromodes on torus.}} \exp\left(-\frac{1}{4} R_{ijkl} \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l\right)$$

$$\begin{cases} = 0 & n: \text{odd} \\ = \dfrac{(-1)^{n/2}}{2^n (n/2)!\, \pi^{n/2}} \cdot \displaystyle\int d^n x\, \epsilon^{i_1 \cdots i_n} R_{i_1 i_2} \wedge \cdots \wedge R_{i_{n-1} i_n}. \end{cases}$$

(e.g. $n=4$

$$\frac{1}{32\pi^2} \int d^4 x\, \epsilon^{abcd} R_{ab} \wedge R_{cd}.$$)

(B) $\Leftrightarrow$ (C) : Gauss-Bonnet formula.

~~Compact~~ $\boxed{\text{Twisted Witten indices}}$ [c.f. chern $\sum_\alpha e^{x_\alpha}$]

$$L(M) = \prod_\alpha \frac{x_\alpha}{\tanh x_\alpha} \quad : \text{Hirzebruch signature}$$

$$\hat{A}(M) = \prod_\alpha \frac{x_\alpha/2}{\sinh x_\alpha/2} \quad : \text{Dirac genus}$$