Superconformal Symmetry and Geometry of Ricci-Flat Kähler Manifold

Hirosi Ooguri

School of Natural Sciences, The Institute for Advanced Study
Olden Lane, Princeton
New Jersey 08540, U.S.A. *
and
Department of Physics, University of Tokyo
7-3-1 Hongo, Bunkyo-ku
Tokyo 113, Japan

*Present address

ABSTRACT

A supersymmetric non-linear σ-model provide us a tool for studying the dynamics of superstrings on a compactified space. If the compactified space is a Ricci-flat Kähler manifold, the non-linear σ-model has extended superconformal symmetry and the partition function of the model is expressed in terms of characters of the superconformal algebra. The partition function must be modular invariant and this condition gives strong constraints on the spectrum of the model. These constraints are intimately related to geometry of the compactified space. Their implications to particle spectra of the compactified heterotic string theory are also discussed.

1. INTRODUCTION

The year of 1984 was the begining of the superstring age. Soon after the discovery of the anomaly cancellation mechanism for the ten-dimensional supergravity and super Yang-Milles theory with gauge group $SO(32)$ or $E_8 \otimes E_8$[1], the heterotic superstring theory with $E_8 \otimes E_8$ gauge symmetry was constructed[2], whose low energy effective field theory was shown to have a solution with four-dimensional flat spacetime and with realistic particle spectra and gauge symmetries[3]. In the following few years, most of the phenomenological analysis of solutions to the superstring theory had been made based on the low energy effective field theory except for some models like orbifolds, which can be constructed from free fields on the string world-sheet. To get reliable results, however, analysis directly based on superstring on compactified space are desirable. In recent years advances in technologies of conformal field theory have made it possible to obtain exact results on compactifications of extra dimensions in the superstring theory.

To build a unification model from superstrings, one has to understand the dynamics of superstrings on a compactified space. Supersymmetric non-linear σ-model on a two-dimesional world-sheet provide us a tool for studying this dynamics. In this paper I will forcus on the extended superconformal symmetry in the non-linear σ-model and derive some exact results concerning the particle spectra of compactified superstring theory. In section 2, I will review on the $N = 2$ superconformal invariance. This symmetry is common to all the non-linear σ-models we will consider here[4], and is intimately related to the spacetime supersymmetry in uncompactified dimensions[5][6][7]. If we consider compactification down to six dimensions, the non-linear σ-model will acquire the $N = 4$ superconformal symmetry. Then its partition function is expressed in terms of characters of the $N = 4$ superconformal algebra. If we require the model to be invariant under all possible conformal transformations including those not continuously connected to the identity, the partition function must be invariant under modular transformations and this condition gives strong restrinctions on the partition function. We discuss the general properties of a modular invariant partition function in section 3. There we will see that informations on the topology of the target space (complex two-dimensional K_3 surface) are naturally encoded in the partition function. Once the partition function for a non-linear σ-model is obtained, one can read off the particle spectra of the compactified heterotic string theory, and this procedure is described in section 4. There we will see the connection between the extended superconformal invariance and the spacetime supersymmetry. Though compactification down to six dimensions is not realistic from the phenomenological point of view, I devote most parts of this paper to it. This is because in this case we have sufficient amount of knowledge of both the symmetry of the non-linear σ-model and the geometry of its target space, and we can appreciate its general structure rather in detail. It will provide a good exercise for us before attacking compactification down to four dimensions. In the last section, I describe some result on the latter problem.

In the last three years we have observed the intriguing connection between the conformal symmetry and the complex geometry of the string world-sheet (a Riemann surface) and it has been proved important in understanding multi-loop quantum effects in the string theory. I hope the studies of the relation between the superconformal symmetry and the geometry of a Ricci-flat Kähler manifold will also uncover profound structure in the superstring theory.

This paper is based on the work[8] with Professor Tohru Eguchi, Doctor Anne Taormina and Doctor Sung-Kil Yang. I would like to thank them for collaboration and discussions on this subject. I am grateful to Professor Takeo Inami for careful reading of the manuscript. This work is supported in part by Department of Energy of the United States of America under contract number DE-AC02-76ER02220.

2. $N = 2$ Superconformal Symmetry

The target space of a two-dimensional supersymmetric non-linear σ-model describing a compactified theory with spacetime supersymmetry has to be a Ricci-flat Kähler manifold[3]. The non-linear σ-model of this class has $N = 2$ supersymmetry[4]. In this section, I will explain how this symmetry comes about and will review some general properties of theories with this symmetry.

Supposе that ten-dimensional flat spacetime for the heterotic superstring theory is compactified down to four dimensions by curling up its extra six dimensions.

$$\boldsymbol{R}^{9,1} \longrightarrow M_6 \times \boldsymbol{R}^{3,1}$$

Each field configuration of M_6 will gives different effective theory in four dimensions. Thus among symmetries of the original ten-dimensional theory, only those preserving the field configuration on M_6 will remain unbroken after compactification. Conversely if we want some symmetry to be observable in four dimensions, the field configuration cannot be arbitrary. It was shown by Candelas, Horowitz, Strominger and Witten[3] using the low-energy effective field theory that the spacetime supersymmetry in four dimensions requires M_6 to be Ricci-flat and Kähler.

These geometrical terms mentioned above mean the following. The curvature tensor $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ has four indices; the first two define an infinitesimal parallelogram in M_6, and the latter two describe how a tangent vector is rotated when it is parallel-transported around the parallelogram, i.e. $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ is a matrix of $SO(6)$ concerning the latter indices ρ and σ. When M_6 is Kähler, we can choose complex coordinates on it in such a way that their holomorphic and anti-holomorphic indices are not mixed when a vector is parallel-transported. This is equivalent to saying that the curvature tensor is now a matrix of $U(3)$, a subgroup of $SO(6)$. Furthermore Ricci-flatness means that a trace of $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ with respect to the latter two indices vanishes. Thus the curvature tensor belongs to an $SU(3)$ subgroup of $U(3)$.

$$SO(6) \xrightarrow{\text{Kähler}} U(3) \xrightarrow{\text{Ricci-flat}} SU(3)$$

Dynamics of superstrings on a curved space is described by a supersymmtric non-linear σ-model on the world-sheet. Its dynamical variables are two-dimensiona bosons $X^i(z,\bar{z})$ and $X^{\bar{j}}(z,\bar{z})$, which geometrically is an embedding map of the world-sheet into the target space M_6, and fermions $\psi^i_\pm$ and $\psi^{\bar{j}}_\pm$ carrying an index of a tangent vector on M_6 $(\gamma_5\psi_\pm = \pm\psi_\pm)$.

In the following, an important role is played by a world-sheet current defined by

$$J(z,\bar{z}) = g_{i,\bar{j}}(X)\psi^i_+\psi^{\bar{j}}_+ \ ,$$

where $g_{i,\bar{j}}$ is a metric on the target space. The fermions ψ^i form a section of a vector bundle over the world sheet defined by pulling back the tangent bundle on the target space by X^i. Then the standard computation of chiral anomaly leads to the observation that $\partial_{\bar{z}}J$ is proportional to the first Chern character $R_{i\bar{j}}\partial_z X^i \partial_{\bar{z}} X^{\bar{j}}$ of the vector bundle[9]. When the target space is Ricci-flat, we obtain $\partial_{\bar{z}}J = 0$, i.e. J is a holomorphic current on the world sheet. This phenomenon can be understood intuitively as follows. If M_6 were flat, all the fermions would be free and they would make $SO(6)$ current algebra. In the presence of the curvature, this symmetry is in general broken. Now $SO(6)$ is locally isomorphic to $SU(4)$ and it has $U(1) \otimes SU(3)$ as a sub-group. In the Ricci-flat Kähler manifold, the curvature generates $SU(3)$ rotation of a tangent vector. Thus the $SU(3)$ part of the $SO(6)$ current algebra may be broken by the curvature effect, however, the $U(1)$ part remains unbroken. The holomorphic current J defined above generates this $U(1)$ current algebra.

It has been know that the supersymmetric non-linear σ-model for M_6 is ultra-violet finite in lower orders of perturbation theory[10][11]. Though there appear divergences from the fourth order[12], they can be cancelled by adding counter terms order by order in perturbation in such a way that the fixed point struncture of the theory is not affected[13]. It has also been argued that this feature persists non-perturbatively[14][15]. In the following we assume that the system we are considering has $N = 1$ superconformal symmetry.

Combining $N = 1$ superconformal symmetry with $U(1)$ current algebra gives $N = 2$ superconformal symmetry. In addition to the Virasoro generators L_n and the U(1) current generators J_n $(n \in \boldsymbol{Z})$, supercurrents come in pairs, G_r and $\overline{G}_r$, with opposite $U(1)$ charges;

$$[J_n, G_r] = + \; G_{r+n} \; , \quad [J_n, \overline{G}_r] = - \; \overline{G}_{r+n} \; ,$$

where $r \in \boldsymbol{Z} + 1/2$ for the Neveu-Schwarz (NS) algebra and $r \in \boldsymbol{Z}$ for the Ramond (R) algebra. A remarkable feature of $N = 2$ superconformal algebra is that there is an isomorphism between the NS and the R algebras[5][16]. If L_n, J_n and $G_r, \overline{G}_r$ are the generators of the $N = 2$ algebra, one may define a new set of operators as

$$\begin{aligned} L'_n &= L_n + \epsilon \; J_n + \frac{c}{6} \; \epsilon^2 \; \delta_{n,0} \\ J'_n &= J_n + \frac{c}{3} \; \epsilon \; \delta_{n,0} \\ G'_r &= G_{r+\epsilon}, \quad \overline{G}_r{}' = \overline{G}_{r-\epsilon} \; , \end{aligned} \tag{1}$$

and, for an arbitrary value of ϵ, they satisfy the same commutation relations as the original ones. Here c is the central charge of the $N = 2$ algebra, which is equal to $3 \times$ (the number of complex dimensions of the target space). Note that, in the above, modding of supercurrents G_r and $\overline{G}_r$ is shifted by ϵ. In particular if we take $\epsilon = 1/2$, the R algebra and the NS algebra are related; the R algebra can be realized in a representation space of the NS algebra and vice versa. This phenomenon is called the *spectral flow.*

The spectral flow of the $N = 2$ symmetry can be neatly illustrated by using the character of a representation of the NS (R) algebra defined as

$$ch_{NS(R)}(\tau, \theta) = trace_{NS(R)}(q^{L_0 - c/24} e^{i\theta J_0}) \tag{2}$$

$$(q = e^{2\pi i \tau}) \; ,$$

where the trace is taken over the representation space. It keeps track of degeneracy of states at each energy level and for each $U(1)$ charge.

Characters for the NS algebra, $ch_{NS}(\tau, \theta)$, are basic building blocks to express the partition function of the supersymmetric non-linear σ-model on a two-dimensional torus, where the fermions ψ^i $(\psi^{\bar{i}})$ are anti-periodic in both of its homology cycles. Now if we shift the argument θ in the NS character by $\pi\tau$, the exponent of q in the trace becomes $L_0 + \frac{1}{2}J_0$ instead of L_0. Employing the isomorphism of the NS and R algebras (1), we obtain the formula

$$ch_{NS}(\tau, \theta + \pi\tau) = ch_R(\tau, \theta) \cdot q^{-c/24} e^{-i\frac{c}{6}\theta} \ , \tag{3}$$

where the trace in the right hand side is taken over the representation space of the R algebra into which the spectrum of the NS representation in flows. These R characters give the partition function of the theory on a torus when the fermions are anti-periodic in one homological cycle and periodic in the other. The spectral flow property of the $N = 2$ characters is fundamental to spacetime supersymmetry in the compactified string theory, though it is not sufficient for this as we shall see below.

These characters have other properties which will prove important in the following discussion. Let us compare the following two equations;

$$\begin{aligned} ch_{NS}(\tau + 1, \theta) &= trace\left(e^{2\pi i(L_0 - c/24)} q^{L_0 - c/24} e^{i\theta J_0}\right) \ , \\ ch_{NS}(\tau, \theta + \pi) &= trace\left(e^{i\pi J_0} q^{L_0 - c/24} e^{i\theta J_0}\right) \ . \end{aligned} \tag{4}$$

It is easy to convince oneself that the identity $(-1)^{2(L_0 - h)} = (-1)^{J_0 - Q}$ holds on an irreducible representation space of the NS algebra, where h and Q are highests weights for L_0 and J_0 respectively. This is because the generators of the NS algebra either commute or anti-commte with both $(-1)^{2(L_0-h)}$ and $(-1)^{J_0 - Q}$; those which commute with them are L_n's and J_n's and those which anti-commute with them are G_r $(\overline{G}_r)$'s. Then by comparing the above two equations, we obtain

$$ch_{NS}(\tau + 1, \theta) = ch_{NS}(\tau, \theta + \pi) \cdot e^{2\pi i(h - c/24 - Q/2)}. \tag{5}$$

Finally ler us see what happens if we put $\theta = \pi + \pi\tau$. From eq.(3), we get

$$\begin{aligned} ch_{NS}(\tau, \pi + \pi\tau) &= ch_R(\tau, \pi) q^{-c/24} e^{-i\frac{c}{6}\pi} \\ &= trace_R((-1)^{J_0} q^{L_0 - c/24}) q^{-c/24} e^{-i\frac{c}{6}\pi}. \end{aligned} \tag{6}$$

Since J_0 counts the fermion number of a state, the right hand sind in the above gives the Witten index[17] of the representation.

Due to the isomorphism in the $N = 2$ algebra, informations on both NS and R representations are simultaneously encoded in the single NS character $ch_{NS}(\tau, \theta)$, and they can be read off by shifting the variable θ in directions, $\pi\tau$, π and $\pi + \pi\tau$.

$$\begin{aligned} ch_{NS}(\tau, \theta + \pi\tau) &= ch_R(\tau, \theta) q^{-c/24} e^{-i\frac{c}{6}\theta} \\ ch_{NS}(\tau, \theta + \pi) &= ch_{NS}(\tau + 1, \theta) e^{-2\pi i (h - c/24 - Q/2)} \\ ch_{NS}(\tau, \pi + \pi\tau) &= (\textit{Witten index}\ ) \times (-q^{-c/24}) \end{aligned} \tag{7}$$

3. K_3 SURFACE AND $N = 4$ SUPERCONFORMAL SYMMETRY

3.1 $N = 4$ Superconformal Symmetry

$N = 2$ superconformal symmetry in a supersymmetric non-linear σ-model is a commom feature for any Kähler manifold with a Ricci-flat metric. When the target space has two complex dimensions, there appears additional chiral symmetry, which makes it possible to examine the spectra of the corresponding non-linear σ-model in more detail.

In the previous section, we have seen how the $U(1)$ current algbra emerges in the non-linear σ-model. Let us repeat the discussion there in the case of complex two-dimensional manifold. If the manifold were flat, the non-linear σ-model would have $SO(4)$ current algebra. $SO(4)$ is locally isomorphic to $SU(2) \otimes SU(2)$.

A curvature of the target space serves to break this current algebra. However, for a Ricci-flat Kähler manifold, one of these $SU(2)$ parts is kept intact. Thus in this case the non-linear σ-model has $SU(2)$ current algebra including the $U(1)$ as its sub-algebra.

$$SO(4) \rightarrow SU(2)$$

It is known that two-dimensional Ricci-flat Kähler manifolds are devided into two topological classes; K_3 surfaces with non-trivial $SU(2)$ curvatures and flat tori. Every K_3 surface has a unique holomorphic two-form, $\epsilon_{ij}(X)dX^i \wedge dX^j$, which we can employ to define currents on the world sheet.

$$J^+(z) = \epsilon_{ij}\psi_+^i\psi_+^j \ , \quad J^-(z) = \epsilon_{\bar{i},\bar{j}}\psi_+^{\bar{i}}\psi_+^{\bar{j}} \tag{8}$$

These two currents, combined with the original $U(1)$ current $J = g_{i\bar{j}}\psi_+^i\psi_+^{\bar{j}}$, make the level-1 $SU(2)$ current algebra, and the $N=2$ superconformal symmetry of the non-linear σ-model is enhanced to the $N=4$ superconformal symmetry with $c = 3 \times 2 = 6$.

Since the $N = 4$ algebra contains the $SU(2)$ current algebra, its representation space is decomposed into the sum of $SU(2)$ representations and its character, $ch^{N=4}(\tau,\theta)$, can be expanded in terms of the $SU(2)$ characters. The level-1 $SU(2)$ current algebra has two unitary representations, one with a $SU(2)$ singlet ground state ($l = 0$) and the other with a doublet ground state ($l = 1$), and their characters are given by

$$\begin{aligned} ch_{l=0}^{SU(2)} &= \frac{1}{\eta(\tau)}\sum_{n\in Z} q^{n^2}e^{i2n\theta} \\ &= \frac{1}{2ch_{l=0}^{SU(2)}(\tau,\theta=0)}\left[\left(\frac{\vartheta_3(\tau,\theta)}{\eta(\tau)}\right)^2 + \left(\frac{\vartheta_4(\tau,\theta)}{\eta(\tau)}\right)^2\right] \ , \\ ch_{l=1}^{SU(2)} &= \frac{1}{\eta(\tau)}\sum_{n\in Z} q^{(n+1/2)^2}e^{i(2n+1)\theta} \\ &= \frac{1}{2ch_{l=1}^{SU(2)}(\tau,\theta=0)}\left[\left(\frac{\vartheta_3(\tau,\theta)}{\eta(\tau)}\right)^2 - \left(\frac{\vartheta_4(\tau,\theta)}{\eta(\tau)}\right)^2\right] \end{aligned} \tag{9}$$

Here $\eta(\tau)$ is the Dedekind η-function, and ϑ_3 and ϑ_4 are elliptic ϑ-functions. Then a character for the NS sector of the $N=4$ algebra is expanded in terms of them as

$$\begin{aligned} ch_{NS}^{N=4}(\tau,\theta) &= c_1(\tau)\, ch_{l=0}^{SU(2)}(\tau,\theta) + c_2(\tau)\, ch_{l=1}^{SU(2)}(\tau,\theta) \\ &= \tilde{c}_1(\tau)\left(\frac{\vartheta_3(\tau,\theta)}{\eta(\tau)}\right)^2 + \tilde{c}_2(\tau)\left(\frac{\vartheta_4(\tau,\theta)}{\eta(\tau)}\right)^2 . \end{aligned} \tag{10}$$

Here the branching coefficients $c_i(\tau)$ $(\tilde{c}_i(\tau))$ $(i=1,2)$ depend only on τ.

Since the $N=4$ superconformal algebra also contains the $N=2$ algebra, the character $ch_{NS}^{N=4}(\tau,\theta)$ should satisfy the similar equations as (7). The equation

$$ch_{NS}^{N=4}(\tau,\theta=\pi\tau+\pi) = (\,Witten\ index\,)(-q^{-1/4})$$

gives a relation between $\tilde{c}_1(\tau)$ and $\tilde{c}_2(\tau)$ in eq.(10), and by solving it we obtain

$$ch_{NS}^{N=4}(\tau,\theta) = -(\,Witten\ index\,)\cdot\left(\frac{\vartheta_1(\theta)}{\vartheta_3^{(0)}}\right)^2 + c(\tau)\left(\frac{\vartheta_3(\theta)}{\eta}\right)^2 \tag{11}$$

$$(\vartheta_3^{(0)} = \vartheta_3(\theta=0)).$$

Thus we are left with one unknown function $c(\tau)$.

Unitary representations of the $N=4$ superconformal algebra were studied by Eguchi and Taormina[18]. They classfied the representations into three. First, for an arbitrary positive value of a conformal weight h, there is a unitary representation without Witten index. Its ground state is $SU(2)$ singlet and the character is given by

$$ch_h^{(0)}(\tau,\theta) = q^h\cdot\frac{q^{-1/8}}{\eta}\left(\frac{\vartheta_3(\theta)}{\eta}\right)^2 . \tag{12}$$

One can see that a character of $SO(4)$ current algebra is factored out in the above expression. Thus the $SO(4)$ symmetry remains unbroken in these representations. There are two representations with non-vanishing Witten indices.

The one is with a singlet ground state of $h = 0$, and the other is with a doublet ground state of $h = 1/4$. Their characters are

$$\begin{aligned} ch_{l=0}(\tau,\theta) &= +2 \cdot \left(\frac{\vartheta_1(\theta)}{\vartheta_3^{(0)}}\right)^2 + \left(\frac{q^{-1/8}}{\eta} - 2h_3(\tau)\right) \cdot \left(\frac{\vartheta_3(\theta)}{\eta}\right)^2 , \\ ch_{l=1}(\tau,\theta) &= -1 \cdot \left(\frac{\vartheta_1(\theta)}{\vartheta_3^{(0)}}\right)^2 + h_3(\tau) \cdot \left(\frac{\vartheta_3(\theta)}{\eta}\right)^2 , \end{aligned} \tag{13}$$

where $h_3(\tau)$ is given by

$$h_3(\tau) = \frac{1}{\eta(\tau)\vartheta_3^{(0)}(\tau)} \sum_{m\in Z} \frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}. \tag{14}$$

One can see that $SO(4)$ characters cannot be factored out from the above expressions (13), and the $SO(4)$ current algebra does not act on the representation spaces with non-vanishing Witten indices. The identity

$$ch_{l=0}(\tau,\theta) + 2 \cdot ch_{l=1}(\tau,\theta) = ch_{h=0}^{(0)}(\tau,\theta) \ , \tag{15}$$

which can easily be read off from the above equations, is also relevant in the following discussion.

3.2 Spectrum of Supersymmetric Non-Linear σ-Model

In this section we derive some exact results on spectrum of a supersymmetric non-linear σ-model for K_3 surface.

Consider a torus defined by $\boldsymbol{C}/\Lambda$ where Λ is a lattice generated by 1 and τ. If one put the non-linear σ-model on this torus with anti-periodic boundary conditions for fermions in both homology cycles, its partion function is expanded in terms of the NS characters of the $N = 4$ superconformal algebra and given as

$Z(\theta=\overline{\theta}=0)$ where

$$
\begin{aligned}
Z_{NS}(\theta,\overline{\theta}) = &\ |ch_{l=0}(\theta)|^2 + N_{1,1}|ch_{l=1}(\theta)|^2 \\
&+ \sum_h N_{h,0}\left(ch_h^{(0)}(\theta)\cdot\overline{ch_{l=0}(\theta)} + complex\ conjugate\right) \\
&+ \sum_h N_{h,1}\left(ch_h^{(0)}(\theta)\cdot\overline{ch_{l=1}(\theta)} + complex\ conjugate\right) \\
&+ \sum_{h,\overline{h}} N_{h,\overline{h}}\ ch_h^{(0)}(\theta)\cdot\overline{ch_{\overline{h}}^{(0)}(\theta)}\ .
\end{aligned}
\tag{16}
$$

Here the coefficients $N_{1,1}$, $N_{h,0}$, $N_{h,1}$ and $N_{h,\overline{h}}$ are either zero or positive integers and count multiplicities of the $N=4$ representations in the total Hilbert space of the non-linear σ-model. The only assumptions in the above expression are that the vacuum state, i.e. the highest weight state in $|ch_{l=0}|^2$ appear with multiplicity one and that there is no cross term of the form $ch_{l=0}\cdot\overline{ch_{l=1}}$. The latter is the same as demanding that there is no free fermion in the spectrum. Presence of free fermion would signal that the target space is flat in some direction. Since there are only two topological classes for two-dimensional Ricci-flat Kähler manifold, K_3 and torus, free fermion implies that the target space is a torus, which is not our concern here.

If we consider the $E_8\otimes E_8$ heterotic superstring theory on $K_3\times \boldsymbol{R}^{5,1}$, thes coefficients N are related to the particle spectra to be observed in the uncompactified six dimensions $\boldsymbol{R}^{5,1}$. In the next section, we will consider the simplest compactification scheme where one of the E_8 gauge symmetry is broken down to E_7 upon compactification. Then it will be shown that the coefficient $N_{1,1}$ gives the number of massless spinors in six-dimensions of the $\underline{56}$ representation of E_7 while $N_{h=1,1}+2N_{1,1}$ is the number of E_7 singlet massless spinors. The coefficient $N_{h=1,0}$ counts the number of extra $U(1)$ gauge bosons generated upon compactification.

If one assumes that the heterotic string theory in ten dimensions can be well-approximated by its low energy effective field theory, the massless particle spectra in the six dimensions are estimated as follows[19]. The number of the spinors in $\underline{56}$ of E_7 is given by the number of holomorphic $(1,1)$-forms on K_3. All the K_3 surfaces have the same topological nature, and the number of holomorphic $(1,1)$-forms is always 20. On the other hand, the coefficient $N_{h=1,1}$ related to the number of E_7-singlets is twice the number of deformations of complex structure for holomorphic tangent bundle over K_3 and it is 90.

Now we are going to derive some exact relations among these coefficients by imposing the modular invariance for the partition function. The partition function $Z(\theta = \overline{\theta} = 0)$ as a function of τ should be invariant under the modular transformations which do not change the periodicities of the fermions; $\tau \to \tau + 2$ and $\tau \to -1/\tau$. To appreciate consequences of this condition, it is useful to consider a function $E(\tau)$ defined by

$$E(\tau) \cdot \left(\frac{1}{\eta(\tau)}\right)^4 \left(\frac{\vartheta_3^{(0)}(\tau)}{\eta(\tau)}\right)^2 \equiv Z(\theta = 0, \overline{\theta} = \pi + \pi\overline{\tau}) \cdot (-\overline{q}^{1/4}) \ . \tag{17}$$

Then the modular invariance of Z implies that

$$E(-1/\tau) = \tau^2 E(\tau) \ , \quad E(\tau + 2) = E(\tau) \tag{18}$$

This property determines the function $E(\tau)$ upto a constant multiplier. The constant is fixed by demanding that the vacuum state appears with multiplicity one. Since the Witten index of the $l = 0$ (vacuum) representation is -2, i.e.

$$ch_{l=0}(\tau, \theta = \pi + \pi\tau) = -2 \times (-q^{-1/4}) \ ,$$

we obtain $E(\tau) = -2 + \cdots$ as $q \to 0$. Thus the function $E(\tau)$ is completely fixed as

$$E(\tau) = 2 \cdot \left((\vartheta_2^{(0)})^4 - (\vartheta_4^{(0)})^4)\right) \ .$$

On the other hand, substituting

$$\begin{aligned} ch_{l=0}(\tau, \theta = \pi + \pi\tau) &= -2 \cdot (-q^{-1/4}) \\ ch_{l=1}(\tau, \theta = \pi + \pi\tau) &= +1 \cdot (-q^{-1/4}) \\ ch^{(0)}_{l=0}(\tau, \theta = \pi + \pi\tau) &= 0 \end{aligned} \tag{20}$$

into eq.(17), $E(\tau)$ is related to N's as

$$E(\tau)\cdot\left(\frac{1}{\eta(\tau)}\right)^4\left(\frac{\vartheta_3^{(0)}(\tau)}{\eta(\tau)}\right)^2 = -2\cdot ch_{l=0}(\tau)+N_{1,1}\cdot ch_{l=1}(\tau) + \sum_h (N_{h,1}-2N_{h,0})\cdot ch_h^{(0)}\ . \tag{21}$$

Comparing the above with eq.(19) using the expressions for the characters (12), (13), we obtain

$$\begin{aligned} F(\tau) &\equiv \sum_h (N_{h,1}-2N_{h,0})q^h \\ &= 2+2q^{1/8}\eta(\tau)\left(\frac{(\vartheta_2^{(0)}(\tau))^4-(\vartheta_4^{(0)}(\tau))^4}{\eta(\tau)^4}-(4+N_{1,1})\cdot h_3(\tau)\right)\ . \end{aligned} \tag{22}$$

Now we argue that $N_{h,0}$ and $N_{h,1}$ should vanish unless h is an integer. By shifting τ as $\tau\to\tau+1$ we obtain a torus with the same complex structure, but the periodicity of the fermion is flipped in one homology cycle. The change of the periodicity for one cycle is taken care of by inserting the $U(1)$ current $J(z)$ along its dual cycle and, in this case, we can just shift θ as $\theta\to\theta+\pi$. This procedure does not break the superconformal invariance, and the partition function should remain invariant under the simultaneous operations $\tau\to\tau+1$ and $\theta\to\theta+\pi$. When h is not an integer, $ch_{l=0}\cdot\overline{ch_h^{(0)}}$ and $ch_{l=1}\cdot\overline{ch_h^{(0)}}$ are not invariant under this operation, and that their coefficients $N_{h,0}$ and $N_{h,1}$ in the partition function $Z(\theta,\bar\theta)$ should vanish.

Since the sum in the first line of eq.(22) is over integral h, the fractional powers of q in the second line should cancel among themselves. This fixes $N_{1,1}$ to be 20. Once $N_{1,1}$ is known, the function $F(\tau)$ is determined completely and so are the numbers $N_{h,1}-2N_{h,0}$.

$$\begin{aligned} F(\tau) &= 90q+462q^2+1540q^3+4554q^4+11592q^5 \\ &\quad +27830q^6+61686q^7+131100q^8+\cdots \end{aligned} \tag{23}$$

Especially $N_{h=1,1} - 2N_{1,1}$ is fixed to be 90. In fact $E(\tau)$ in the above gives the elliptic genus[20][21][22] for K_3 surface, and all the expansion coefficients are topological invariants.

If q^h in $F(\tau)$ had a negative coefficient for some value of h, the corresponding $N_{h,0}$ would be non-vanishing. If this were the case, there should be a holomorphic field of dimension h, and the symmetry of the system could not be just the $N = 4$ symmetry but always larger than that. I have computed the expansion coefficients of $F(\tau)$ to the order of q^{50} and found that they are all positive and exponentially increasing. One can also examine the asymptotic behaviour of $F(\tau)$ as $\tau \to 0$, i.e. $q \to 1$. Using the modular transformation property of the ϑ-functions and the Mordell's formula[23] for $h_3(\tau)$,

$$h_3(-1/\tau) = -h_3(\tau) + \frac{1}{\eta(\tau)} \int_{-\infty}^{\infty} d\alpha \frac{q^{\alpha^2/2}}{2cosh\pi\alpha} \ ,$$

we obtain

$$F(-1/\tau) = 2 + \sqrt{-i\tau}\ \tilde{q}^{1/8} \left(q^{-1/8} \cdot (2 - F(\tau)) + 24 \cdot \int_{-\infty}^{\infty} d\alpha \frac{q^{\alpha^2/2}}{2cosh\pi\alpha} \right) ,$$

where $\tilde{q} = e^{-2\pi i/\tau}$. Taking the limit $\tau \to +i\infty \quad (\tilde{q} \to 1)$, the above gives

$$\begin{aligned} F(-1/\tau) &= \sum_h (N_{h,1} - 2N_{h,0}) \cdot \tilde{q}^h \\ &\longrightarrow 2\sqrt{-i\tau}q^{-1/8} - 12 + \cdots . \end{aligned} \tag{24}$$

This observation seems to imply that the q-expansion coefficients of $F(\tau)$ are all positive and the symmetry of the generic non-linear σ-model is just the $N = 4$ superconformal symmetry, though I have no rigorous proof for it. Of course this does not exclude the possibility that the symmetry of the system is enhanced in some particular case, such as extra $U(1)$ gauge symmetries in orbifold models.

Now that we have determined all the numbers $(N_{h,1} - 2N_{h,0})$, we can write the partition function as

$$\begin{aligned} Z_{NS} = & \, |ch_{l=0}|^2 + 20|ch_{l=1}|^2 \\ & + (90 + 2x_1) \; ch^{(0)}_{h=1} \cdot \overline{ch_{l=1}} + x_1 \; ch^{(0)}_{h=1} \cdot \overline{ch_{l=0}} \\ & + (462 + 2x_2) \; ch^{(0)}_{h=2} \cdot \overline{ch_{l=1}} + x_2 \; ch^{(0)}_{h=2} \cdot \overline{ch_{l=0}} \\ & + \cdots + complex \; conjugate \\ & \quad + \sum_{h,\bar{h}>0} N_{h,\bar{h}} \; ch^{(0)}_{h} \cdot \overline{ch^{(0)}_{\bar{h}}} \; . \end{aligned} \tag{25}$$

The coefficients x_i's are not determined from the above consideration, and depend on the detail of the model. If one employ eq.(15), the above can be expressed in a more compact form as

$$\begin{aligned} Z_{NS} = & \, |ch_{l=0}|^2 + 20|ch_{l=1}|^2 \\ & + \Big(\, 90ch^{(0)}_{h=1} + 462ch^{(0)}_{h=2} + 1540ch^{(0)}_{h=3} + 4554ch^{(0)}_{h=4} + \cdots \Big) \cdot \overline{ch_{l=1}} \\ & + complex \; conjugate + \sum_{h,\bar{h}\geq 0} N_{h,\bar{h}} \; ch^{(0)}_{h} \cdot \overline{ch^{(0)}_{\bar{h}}} \; . \end{aligned} \tag{26}$$

Thus the partition function of the non-linear σ-model is clearly separeted into two parts; those which are universal to all the models with the $N = 4$ symmetry carrying the topological informations of K_3 surface, and those depending on details of each model.

In this subsection, we have seen some general properties of the modular invariant combination of the $N = 4$ characters. By imposing the modular invariance, we obtain infinitely many relations among the coefficients of the modular invariants and they reproduce topological invariants of K_3 surface. An arbitrary K_3 surface will give a ultraviolet-finite supersymmetric non-linear σ-model. The result in this section seems to suggest that, conversely for an arbitrary modular invariant of the $N = 4$ characters, there is a geometry of K_3 surface.

3.3 Examples

There are several models with $N = 4$ superconformal symmetry for which we can compute a partition function explicitly.

(1) Orbifolds[24]

There are four possible types of orbifold limits of K_3 surface given by dividing the product of complex tori $T_1 \times T_2$ by the Z_K action $(K = 2,3,4,6)$[25].

$$z_1 \rightarrow e^{2\pi i/K} z_1 \ , \quad z_2 \rightarrow e^{-2\pi i/K} z_2$$

Here z_i is the coordinate of the torus T_i. The partition function for the supersymmetric non-linear σ-model on the orbifold is given by

$$\begin{aligned} Z_{orbifold}(\theta,\bar{\theta}) = & \frac{1}{K}\frac{1}{|\eta(\tau)|^8}\sum_{\vec{w_R},\vec{w_L}} q^{\frac{1}{2}\vec{w_R}^2}\bar{q}^{\frac{1}{2}\vec{w_L}^2}\left|\frac{\vartheta_3(\tau,\theta)}{\eta(\tau)}\right|^4 \\ & + \sum_{0\le r,s\le K-1,\ (r,s)\neq(0,0)} n_{r,s}\left|\frac{\vartheta_3(\tau,\theta+\frac{s+r\tau}{K})\vartheta_3(\tau,\theta-\frac{s+r\tau}{K})}{\vartheta_1(\frac{s+r\tau}{K})^2}\right|^2 , \end{aligned} \tag{27}$$

where $\vec{w_R}$ and $\vec{w_L}$ are weight vectors of a real four-dimensional lattice determined by the shape of the torus $T_1 \times T_2$, and the coefficients $n_{r,s}$ are defined by

$$n_{0,s} = \frac{1}{K}\left(2sin\frac{\pi s}{K}\right)^4$$

and the symmetry relations

$$n_{r,s} = n_{r,s+r} \ , \quad n_{r,s} = n_{s,K-r} \ .$$

It is straightforward to see that the above partition function can be written in the form of eq.(26). In fact

$$\begin{aligned} Z_{orbifold}(\theta=0,\bar{\theta}=\pi+\pi\tau)\cdot(-\bar{q}^{1/4}) &= -\sum_{r,s} n_{r,s}\left(\frac{\vartheta_3(\tau,\frac{s+r\tau}{K})}{\vartheta_1(\frac{s+r\tau}{K})}\right)^2 , \\ &= 2\cdot\left((\vartheta_2)^4-(\vartheta_4)^4\right)\cdot\left(\frac{1}{\eta}\right)^4\left(\frac{\vartheta_3^{(0)}}{\eta}\right)^2 , \end{aligned} \tag{28}$$

and the elliptic genus $E(\tau)$ is correctly reproduced.

(2) Gepner's model[26]

Unitary representations for the $N=2$ superconformal algebra has been worked out by various authors[4][27][28] and, for the discrete series with $c=\frac{3k}{k+2}$ $(k=1,2,3,...)$, modular invariant combinations of their characters were classified[29]. By exploiting these modular invariants, Gepner[26] constructed partition functions which will correspond to a class of K_3 surfaces. Since their central charge c is six, he considered a tensor product of $N=2$ representations so that their central charges add up to six.

$$c=\sum_{i=1}^{M}\frac{3k_i}{k_i+2}=6$$

A unitary representation for the $N=2$ algebra in the descrete series is labeled by two indices, l and m $(l=0,1,...,k,\ m=-l,-l+2,...,l)$, and in the NS sector its conformal weight h and $U(1)$ charge Q are given by

$$h=\frac{l(l+2)-m^2}{4(k+2)}\ ,\quad Q=\frac{m}{k+2}\ .$$

Let us denote its character by $ch_{l,m}^{(k)}(\tau,\theta)$. Basic building blocks of a Gepner's model are products of these characters;

$$\prod_{i=1}^{M}ch_{l_i,m_i}^{(k_i)}(\tau,\theta)\ . \tag{29}$$

Since modular invariant combinations of $ch_{l,m}^{(k)}$ are know, one can readily construct modular invariants of these products (29).

So far we have considered characters of the $N=2$ algebra. The $N=4$ algebra contains the level-1 $SU(2)$ current algebra as well as the $N=2$ algebra and we must take it into account. The $N=4$ characters $ch^{N=4}(\tau,\theta)$ are expanded in terms of the $SU(2)$ characters $ch_l^{SU(2)}(\tau,\theta)$. As one can see from eq.(9), the $SU(2)$ characters are periodic in θ as

$$
\begin{aligned}
ch_l^{SU(2)}(\tau,\theta+2\pi) &= ch_l^{SU(2)}(\tau,\theta) \\
ch_l^{SU(2)}(\tau,\theta+2\pi\tau) &= ch_l^{SU(2)}(\tau,\theta)\cdot q^{-1}e^{-2i\theta}
\end{aligned}
\tag{30}
$$

and so are the $N=4$ characters. On the other hand, the $N=2$ characters for descrete series do not satisfy these properties.

$$
\begin{aligned}
ch_{l,m}^{(k)}(\tau,\theta+2\pi) &= e^{2\pi i\frac{m}{k+2}}\, ch_{l,m}^{(k)}(\tau,\theta) \\
ch_{l,m}^{(k)}(\tau,\theta+2\pi\tau) &= ch_{l,m-2}^{(k)}(\tau,\theta)\cdot q^{-\frac{c}{6}}e^{-i\frac{c}{3}\theta}
\end{aligned}
\tag{31}
$$

Thus for the $SU(2)$ current algebra to act properly on the model, we must impose the charge integrability condition

$$
\sum_i \frac{m_i}{k_i+2} = integer\ , \tag{32}
$$

and sum over the orbit generated by the shift $\theta\to\theta+2\pi\tau$,

$$
\sum_{\beta=0}^{k+1} ch_{l_1,m_1-2\beta}^{(k_1)}\cdots ch_{l_M,m_M-2\beta}^{(k_M)}\ , \tag{33}
$$

where $(k+2)$ is the least common multiplier of $(k_1+2),...,(k_M+2)$, and $ch_{l,m}^{(k)}$ for $|m|\geq l$ is defined by

$$
ch_{l,m}^{(k)} = ch_{k-l,k-m}^{(k)} = ch_{l,m+2(k+2)}^{(k)}\ .
$$

Conversely the orbit of the shift $\theta\to\theta+2\pi\tau$ constructed as in the above can be expanded in terms of the $SU(2)$ characters and the $N=4$ symmetry acts properly on it.

If the charge integrality condition (32) is met, all the terms in the orbit (33) transform in the same way under the modular transformation, $\tau\to-1/\tau$. Also the charge integral orbits transform among each other under the modular tranformation. Thus one can obtain the modular invariants of the charge integral orbits from the modular invariants of the $N = 2$ discrete characters. This construction is quite similar in its spirit to that of an orbifold model. The $N = 2$ sub-system with $c = \frac{3k_i}{k_i+2}$ has Z_{k_i+2} symmetry, $(l, m) \to (l, m+2)$, and their tensor product has $Z_{k_1+2} \otimes \cdots \otimes Z_{k_M+2}$. Gepner's model is given by twisting the simple tensor product of these sub-systems by the action of the diagonalZ_{k+2}.

We have examined the partition functions of Gepner's models explicitly, and found that some of them coincide with those of particular orbifold models[8]. Recently Martinec[30], and Vafa and Warner[31] made an interesting observation on these models concering their connection to algebraic geometry of the target manifolds, but the significance of their observation is yet to be explored.

4. Heterotic String Theory on $R^{5,1} \times K_3$

In section 3, we have studied the sesquilinear form of the $N = 4$ characters

$$\begin{aligned} Z_{NS}(\theta,\overline{\theta}) = & |ch_{l=0}(\theta)|^2 + N_{1,1}|ch_{l=1}(\theta)|^2 \\ & + \sum_h N_{h,0}\left(ch_h^{(0)}(\theta)\cdot\overline{ch_{l=0}(\theta)} + \ complex\ conjugate\right) \\ & + \sum_h N_{h,1}\left(ch_h^{(0)}(\theta)\cdot\overline{ch_{l=1}(\theta)} + \ complex\ conjugate\right) \\ & + \sum_{h,\overline{h}} N_{h,\overline{h}}\ ch_h^{(0)}(\theta)\cdot\overline{ch_{\overline{h}}^{(0)}(\theta)}\ , \end{aligned} \tag{34}$$

which is invariant under modular transformations, $\tau \to -1/\tau$ and $\tau \to \tau + 1$, $\theta \to \theta + \pi$. From the above modular invariant, we are going to derive a partition function of the $E_8 \otimes E_8$ heterotic string theory compactified down to six dimensions.

The partition function of the heterotic string theory in flat ten dimensions is written as

$$Z^{D=10} = \left(\frac{1}{Im\ \tau}\right)^4 \mathcal{R}^{D=10}(\tau;\theta_1,...,\theta_4)\cdot\overline{\mathcal{L}^{D=10}(\tau;\varphi_1,...,\varphi_{16})}\ , \tag{35}$$

where

$$\begin{aligned}
\mathcal{R}^{D=10} &= \frac{1}{2}\left(\frac{1}{\eta}\right)^8 \left(\prod_{a=1}^{4}\frac{\vartheta_3(\theta_a)}{\eta} - \prod_{a=1}^{4}\frac{\vartheta_4(\theta_a)}{\eta} - \prod_{a=1}^{4}\frac{\vartheta_2(\theta_a)}{\eta} + \prod_{a=1}^{4}\frac{\vartheta_1(\theta_a)}{\eta}\right) \\
\mathcal{L}^{D=10} &= \left(\frac{1}{\eta}\right)^8 ch^{E_8}(\varphi_1,...,\varphi_8)\cdot ch^{E_8}(\varphi_9,...,\varphi_{16})\ ,
\end{aligned} \tag{36}$$

and

$$ch^{E_8}(\varphi_1,...,\varphi_8) = \frac{1}{2}\left(\prod_{b=1}^{8}\frac{\vartheta_3(\varphi_b)}{\eta} + \prod_{b=1}^{8}\frac{\vartheta_4(\varphi_b)}{\eta} + \prod_{b=1}^{8}\frac{\vartheta_2(\varphi_b)}{\eta} + \prod_{b=1}^{8}\frac{\vartheta_1(\varphi_b)}{\eta}\right)\ .$$

The partition function $Z^{D=10}$ is fully modular invariant

$$Z^{D=10}(-1/\tau;-\vec{\theta}/\tau,-\vec{\varphi}/\tau) = Z^{D=10}(\tau;\vec{\theta},\vec{\varphi}),$$

$$Z^{D=10}(\tau+1;\vec{\theta},\vec{\varphi}) = Z^{D=10}(\tau;\vec{\theta},\vec{\varphi})\ .$$

Due to the Jacobi identity, $\mathcal{R}^{D=10}(\vec{\theta}=0)=0$, and the particle spectra is space-time supersymmetric.

We are discussing the heterotic string theory in the light-cone gauge, and eight free fermions in the right moving part realize the $SO(8)$ current algebra. Now if we put this theory on $\boldsymbol{R}^{5,1}\times K_3$, $SO(8)$ is decomposed into $SO(4)\otimes SO(4)$ and the latter $SO(4)$ is broken down to $SU(2)$ due to the curvature on K_3 as described in section 3.

$$SO(8)\supset SO(4)\otimes SO(4)\longrightarrow SO(4)\otimes SU(2)$$

Then it is natural to replace the spectrum of four free fermions and four free bosons in $R^{D=10}$ by the $N=4$ superconformal characters $ch_{NS,i}$ (i is an index of a representation) as

$$\mathcal{R}_i = \frac{1}{2}\left(\frac{1}{\eta}\right)^4 \begin{pmatrix} \frac{\vartheta_3(\theta_1)\vartheta_3(\theta_2)}{\eta^2}\cdot ch_{NS,i}(\tau,\theta_3+\theta_4) - \frac{\vartheta_4(\theta_1)\vartheta_4(\theta_2)}{\eta^2}\cdot ch_{\widetilde{NS},i}(\tau,\theta_3+\theta_4) \\ -\frac{\vartheta_2(\theta_1)\vartheta_2(\theta_2)}{\eta^2}\cdot ch_{R,i}(\tau,\theta_3+\theta_4) + \frac{\vartheta_1(\theta_1)\vartheta_1(\theta_2)}{\eta^2}\cdot ch_{\widetilde{R},i}(\tau,\theta_3+\theta_4) \end{pmatrix} \tag{37}$$

where

$$\begin{aligned} ch_{\widetilde{NS},i}(\tau,\theta) &\equiv ch_{NS,i}(\tau,\theta+\pi) \\ ch_{R,i}(\tau,\theta) &\equiv q^{1/4}e^{i\theta}ch_{NS,i}(\tau,\theta+\pi\tau) \\ ch_{\widetilde{R},i}(\tau,\theta) &\equiv ch_{R,i}(\tau,\theta+\pi) = (\,Witten\ index\,)\ . \end{aligned} \tag{38}$$

In summing over spin structures in eq.(37), the relative signs are chosen in such a way that, under the modular transformation $\tau \to -1/\tau$, $\mathcal{R}_i$'s transform in the same way as the $N = 4$ characters $ch_{NS,i}$ and that each $\mathcal{R}_i$ is invariant upto phases under $\tau \to \tau + 1$. They make building blocks for the right moving part of the heterotic string.

Let us now turn to the left moving part. In the analysis based on the low energy effective field theory of the heterotic string, cancellation of anomalies in the compactified theory requires that the integral of a square of the curvature, $R_{ijkl}R^{ijkl}$ is equal to the integral of a square of the Yang-Mills field strength, $trace(F_{ij}F^{ij})$[3][19]. The simplest solution is to embed $SO(4)$ of the tangent bundle of K_3 into one of the E_8 gauge groups and to identity the spin connection to the gauge potential on K_3. This will break one of the E_8 down to E_7.

$$E_8\ \ (\supset SO(12)\otimes SO(4)\ ) \longrightarrow E_7\ \ (\supset SO(12)\otimes SU(2)\ )$$

The right mover of the heterotic string in flat ten dimensions consists of free bosons for eight transverse directions and sixteen free fermions generating the $E_8 \otimes E_8$ gauge symmetry. Thus in the exact spectrum of the compactified theory, the identification of the spin connection to the gauge potential amount to picking up four free bosons and four free fermions from the spectrum and to replacing them with the $N=4$ characters.

$$\mathcal{L}_i = \left(\frac{1}{\eta}\right)^4 \cdot \frac{1}{2} \left(\begin{array}{l} \prod_{b=1}^{6} \frac{\vartheta_3(\varphi_b)}{\eta} \cdot ch_{NS,i}(\tau,\varphi_7+\varphi_8) + \prod_{b=1}^{6} \frac{\vartheta_4(\varphi_b)}{\eta} \cdot ch_{\widetilde{NS},i}(\tau,\varphi_7+\varphi_8) \\ + \prod_{b=1}^{6} \frac{\vartheta_2(\varphi_b)}{\eta} \cdot ch_{R,i}(\tau,\varphi_7+\varphi_8) + \prod_{b=1}^{6} \frac{\vartheta_1(\varphi_b)}{\eta} \cdot ch_{\widetilde{R},i}(\tau,\varphi_7+\varphi_8) \end{array} \right)$$
$$\times\, ch^{E_8}(\tau;\varphi_9,\dots,\varphi_{16}) \tag{39}$$

The signs in eq.(39) are again chosen in such a way that $\mathcal{L}_i$'s transform in the same way as the $N=4$ characters under $\tau \to -1/\tau$ and that each of them invariant upto phases under $\tau \to \tau+1$.

One can now write down a partition function for the heterotic string theory on $\boldsymbol{R}^{5,1} \times K_3$

$$\mathcal{Z}^{D=6} = \left(\frac{1}{Im\ \tau}\right)^2 \cdot \left(\begin{array}{l} \mathcal{R}_{l=0} \cdot \overline{\mathcal{L}_{l=0}} + N_{1,1}\ \mathcal{R}_{l=1} \cdot \overline{\mathcal{L}_{l=1}} \\ + \sum_h N_{h,0} \left(\mathcal{R}_h^{(0)} \cdot \overline{\mathcal{L}_{l=0}} + \ complex\ conjugate \right) \\ + \sum_h N_{h,1} \left(\mathcal{R}_h^{(0)} \cdot \overline{\mathcal{L}_{l=1}} + \ complex\ conjugate \right) \\ \quad + \sum_{h,\bar{h}} N_{h,\bar{h}}\ R_h^{(0)} \cdot \overline{\mathcal{L}_{\bar{h}}^{(0)}} \end{array} \right) . \tag{40}$$

In the partition function for the non-linear σ-model (34) is invariant under modular transformations preserving the spin structure, $\mathcal{Z}^{D=6}$ is invariant under the full modular transformations.

Let us look into details of the spectum (40). As we have seen in the last section, the $N=4$ characters are expanded in terms of the characters of the level-1 $SU(2)$ current algebra.

$$ch_{NS,i}(\tau,\theta) = c_{1,i}(\tau)\, ch_{l=0}^{SU(2)}(\tau,\theta) + c_{2,i}(\tau)\, ch_{l=1}^{SU(2)}(\tau,\theta)\ . \tag{41}$$

Then using a variant of the Jacobi identity, one can show that the particle spectra is supersymmetric, $\mathcal{R}_i(\tau;\vec{\theta}=0)=0$, irrespectively of the branching coefficients

$c_{1,i}$, $c_{2,i}$. Thus the $SU(2)$ current algebra on the world sheet is essential to the spacetime supersymmetry in six dimensions. Concerning the left mover, one obtains

$$\mathcal{L}_i = \left(\frac{1}{\eta}\right)^4 \left(c_{1,i}(\tau) \cdot ch_{\underline{1}}^{E_7} + c_{2,i}(\tau) \cdot ch_{\underline{56}}^{E_7}\right) \cdot ch^{E_8}. \tag{42}$$

Here $ch_{\underline{1}}^{E_7}$ and $ch_{\underline{56}}^{E_7}$ are characters of the E_7 current algebras given by

$$\begin{aligned} ch_{\underline{1}}^{E_7} &= ch_{\underline{1}}^{SO(12)} ch_{l=0}^{SU(2)} + ch_{\underline{32}}^{SO(12)} ch_{l=1}^{SU(2)} \\ ch_{\underline{56}}^{E_7} &= ch_{\underline{12}}^{SO(12)} ch_{l=0}^{SU(2)} + ch_{\underline{32}^*}^{SO(12)} ch_{l=0}^{SU(2)} \ . \end{aligned} \tag{43}$$

The explicit form of the branching coefficients $c_{1,i}$, $c_{2,i}$ can be read off from eqs.(12),(13). From eq.(12) for a character without Witten index, we obtain

$$\mathcal{L}_h^{(0)} = q^h \left(\frac{1}{\eta}\right)^4 ch^{E_8} \cdot ch^{E_8} \ . \tag{44}$$

Thus in this sector, the $E_8 \otimes E_8$ gauge symmetry is not broken. For those with non-vanishing Witten indices, we have

$$\begin{aligned} c_{1,l=0} &= q^{1/24} q^{-1/4} (1 + q^2 + \cdots) \\ c_{2,l=0} &= q^{1/24} (2q + 2q^2 + \cdots) \\ c_{1,l=1} &= q^{1/24} q^{-1/4} (2q + 4q^2 + \cdots) \\ c_{2,l=1} &= q^{1/24} (1 + q + 4q^2 + \cdots) \ . \end{aligned} \tag{45}$$

Substituting them into eqs.(37) and (39), we obtain

$$\begin{aligned} \mathcal{L}_{l=0} &= (vector)^{SO(4)} - (spinor)^{SO(4)} \cdot \underline{2}^{SU(2)} + O(q) \\ \mathcal{L}_{l=1} &= (scalar)^{SO(4)} \cdot \underline{2}^{SU(2)} - (spinor)^{SO(4)} + O(q) \ , \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{l=0} &= q^{-1} + (\underline{133}^{E_7} + \underline{248}^{E_8} + (vector)^{SO(4)}) + O(q) \\ \mathcal{R}_{l=1} &= \underline{56}^{E_7} + \underline{1}^{E_7} \cdot \underline{2}^{SU(2)} + O(q) \ . \end{aligned}$$

Now one can extract the massless particle spectra. The term $\mathcal{R}_{l=0}\cdot\overline{\mathcal{L}}_{l=0}$ gives the supergravity and the $E_7 \otimes E_8$ super Yang-Mills multiplets in six dimensions. On the other hand one scalar multiplet of $\underline{56}$ of E_7 and two E_7-singlet scalar multiplets (making a doublet of $SU(2)$) come from $\mathcal{R}_{l=1}\cdot\overline{\mathcal{L}}_{l=1}$. The cross term of the form $\mathcal{R}^{(0)}_{h=1}\cdot\overline{\mathcal{L}}_{l=0}$ gives an extra-$U(1)$ gauge multiplet and $\mathcal{R}^{(0)}_{h=1}\cdot\overline{\mathcal{L}}_{l=1}$ adds a E_7-singlet scalar multiplet. Thus $N_{1,1}$ is the number of $\underline{56}$ of E_7, and $N_{h=1,0}+2N_{1,1}$ counts the E_7-singlets. The number of extra-$U(1)$ gauge bosons is given by $N_{h=1,0}$.

5. Compactification down to Four Dimensions

We have seen the relation between the superconformal symmetry on the world sheet and the geometry of K_3 surface and examined its implication to the particle spectra of the heterotic string theory on $\boldsymbol{R}^{5,1}\times K_3$. Let us now turn to more realistic case of compactification down to four dimensions.

In the case of K_3 surface, the $N=2$ superconformal symmetry is enhanced to the $N=4$ symmetry. This is due to the holomorphic two-form on K_3, which implies holomorphic currents $J^{\pm}(z)$ on the world sheet. In the case of complex three-dimensional manifold M_6 with non-trivial $SU(3)$ curvature (Calabi-Yau three-fold), we have a unique holomorphic three form $\epsilon_{ijk}dX^i\wedge dX^j\wedge dX^k$, and this also implies the existence of holomorphic fields on the world sheet.

$$\mathcal{X}(z)\equiv\epsilon_{ijk}\psi^i_+\psi^j_+\psi^k_+\ ,\quad \overline{\mathcal{X}}(z)\equiv\epsilon_{ijk}\psi^{\bar{i}}_+\psi^{\bar{j}}_+\psi^{\bar{k}}_+$$

These holomorphic fields have conformal dimension $3/2$ and $U(1)$ charges ± 3. Just as $J^{\pm}$ cobined with the $U(1)$ current J made the level-1 $SU(2)$ current algebra for K_3, we obtain $N=2$ superconformal algebra of $c=1$ from $\mathcal{X}$, $\overline{\mathcal{X}}$ with J and $J\cdot J$ in additional to the original $N=2$ algebra with $c=9$. Thus, for Calabi-Yau three-fold, symmetries of the non-linear σ-model is again larger than $N = 2$.

Repeating the argument in section 3, we see that the building blocks of a partition function for the non-linear σ-model, $ch_{NS}^{3-fold}(\tau,\theta)$, are expanded in terms of the NS characters of the $c = 1$ $N = 2$ superconformal algebra (by construction periodicities of $\mathcal{X}(z)$ are the same as those of the fermions ψ^i). There are three unitary representations for the $c = 1$ $N = 2$ superconformal algebra with the $U(1)$ charges $Q = 0, \pm 1$, and their characters, $f_Q(\tau,\theta)$, are given

$$f_Q(\tau,\theta) = \frac{1}{\eta(\tau)} \sum_{n \in Z} q^{\frac{1}{6}(3n+Q)^2} e^{i(3n+Q)} \ .$$

Then we can write as

$$ch_{NS}^{3-fold}(\tau,\theta) = c_1(\tau) f_{Q=0}(\tau,\theta) + c_2(\tau) f_{Q=+1}(\tau,\theta) + c_3(\tau) f_{Q=-1}(\tau,\theta) \ .$$

By imposing the condition

$$ch_{NS}(\tau, \theta = \pi + \pi\tau) \cdot (-q^{3/8}) = (\,Witten\ index\,) \ ,$$

we obtain

$$\begin{aligned} ch_{NS}^{3-fold}(\tau,\theta) = & -(\,Witten\ index\,) \cdot \frac{1}{2} \left(f_{Q=+1}(\tau,\theta) - f_{Q=-1}(\tau,\theta) \right) \\ & + c(\tau) \cdot f_{Q=0}(\tau,\theta) + d(\tau) \cdot \left(f_{Q=+1}(\tau,\theta) + f_{Q=-1}(\tau,\theta) \right) \ . \end{aligned} \tag{50}$$

Though the representation theory of this symmetry is still to be delevoped** and explicit forms of $c(\tau)$ and $d(\tau)$ are not eleborated yet, we can discuss the consequences of this symmetry from rather simple considerations as follows.

Let us first consider representations with non-vanishing Witten index. By setting $\epsilon = \pm 1/2$ in eq.(1), NS representation *flows* into R representation. In the

**The commutation relations of the extended algebra have receltly been worked out[32].

R sector, the highest weight of L_0'

$$L_0' = L_0 \pm \frac{1}{2} J_0 + \frac{3}{8}$$

in a unitary representation is bounded below by 3/8. For such a representation to have non-vanishing Witten index, the highest weight must hit the bound. Then non-vanishing Witten index requires $h = \pm Q/2$ in the NS sector. Since the θ-dependence of the character $ch_{NS}^{3-fold}(\tau, \theta)$ is dictated by the $c = 1$ $N = 2$ characters $f_Q(\tau, \theta)$, the highest weight for the $U(1)$ charge is either $Q = 0$, $Q = +1$ or $Q = -1$. Thus the candidates are $(h, Q) = (0, 0)$, $(1/2, 1)$ and $(1/2, -1)$. Those without Witten index must lie above these bounds; $(h > 0, Q = 0)$ or $(h > 1/2, Q = \pm 1)$.

The vacuum representation $(h, Q) = (0, 0)$, however, turns out to have a zero Witten index. As one can see from eq.(50), non-vanishing Witten index implies $U(1)$ charge asymmetry in the representation;

$$ch^{3-fold}(\tau, -\theta) \neq ch^{3-fold}(\tau, \theta) \ .$$

The representation with $Q = 0$ is invariant under the charge conjugation and its Witten index must vanish. Thus only those with $(h, Q) = (1/2, 1)$ or $(1/2, -1)$ may have Witten indices.

We can now list up the unitary representations for the Calabi-Yau three-fold.

(1) Representations with Witten index $(h, Q) = (1/2, 1), (1/2, -1)$

Since $f_{Q=\pm 1}(\tau, \theta)$ is expanded as

$$f_{Q=\pm 1}(\tau, \theta) = q^{-\frac{3}{8}} \cdot (q^{\frac{1}{2}} e^{\pm i\theta} + \cdots) \ ,$$

the characters for the Calabi-Yau three-fold $ch^{3-fold}(\tau, \theta)$ with $Q = \pm 1/2$ are written as

$$\begin{aligned} ch_{Q=\pm 1}(\tau, \theta) = & f_{Q=\pm 1}(\tau, \theta) + c_{\pm}(\tau) \cdot f_{Q=0}(\tau, \theta) \\ & + d_{\pm}(\tau) \cdot (f_{Q=1}(\tau, \theta) + f_{Q=-1}(\tau, \theta)) \ . \end{aligned} \tag{51}$$

Witten index for $ch_{Q=\pm1}$ is ±1.

(2) Representations without Witten index

As we have seen all the unitary representation with $Q = 0$ and $h \geq 0$ belong to this class. Also there are representations with $h > 1/2$, when a ground state must form a doublet $Q = \pm1$.

By examining the partition function of Gepner's model for M_6, in which the above symmetry is realized, one can check that the representations listed in the above in fact appear in the spectrum.

As I have illustrated in section 4 in the case of K_3 surface, we can construct the partition function for the heterotic string theory on $\boldsymbol{R}^{3,1} \times M_6$ from a modular invariant sesquilinear form of the characters ch_{NS}^{3-fold}. In this case one of the E_8 gauge symmetry is broken down to E_6, and there appear massless scalar multiplets of $\underline{27}$ and $\underline{27}^*$ of E_6. Their numbers are given from the multiplicities of $(|ch_{Q=+1}|^2 + |ch_{Q=-1}|^2)$ and $(ch_{Q=+1} \cdot \overline{ch}_{Q=-1} + ch_{Q=-1} \cdot \overline{ch}_{Q=+1})$ respectively.

Now a peculiar feature emerges from the following identity.

$$
\begin{aligned}
|ch_{Q=+1}|^2 + |ch_{Q=-1}|^2 = &\left(ch_{Q=+1} \cdot \overline{ch}_{Q=-1} + ch_{Q=-1} \cdot \overline{ch}_{Q=+1}\right) \\
&+ |f_{Q=+1} - f_{Q=-1}|^2
\end{aligned}
\tag{52}
$$

Since $|f_{Q=+1} - f_{Q=-1}|^2$ is invariant under $\tau \to 1/\tau$ and $\tau \to \tau + 2$, interchanging of $(|ch_{Q=+1}|^2 + |ch_{Q=-1}|^2)$ with $(ch_{Q=+1} \cdot \overline{ch}_{Q=-1} + ch_{Q=-1} \cdot \overline{ch}_{Q=+1})$ does not spoil the modular invariance of the partition function. If both of these two partition functions, before and after this operation, correspond to some Calabi-Yau three-folds, Euler number of these three-folds differ by four. Can we understand this as some geometrical operation such as surgery? In the terms of the particle spectra in four dimensions, $\underline{27}$ and $\underline{27}^*$ scalar multiplets of E_6 are interchanged under this operation. This may have some interesting implications in the phenomenology of the superstring compactification.

REFERENCES

1. M. B. Green and J. H. Schwarz, *Phys. Lett.* 149B (1985) 117.

2. D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Phys. Rev. Lett.* 54 (1985) 502, *Nucl. Phys.* B256 (1985) 253, B267 (1986) 75.

3. P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* B258 (1985) 46.

4. W. Boucher, D. Friedan and A. Kent, *Phys. Lett.* 172B (1986) 316.

5. D. Friedan, A. Kent, S. Shenker and E. Witten, *unpublished* (1987).

6. A. Sen, *Nucl. Phys.* B278 (1986) 287, B284 (1987) 423.

7. T. Banks, L. Dixon, D. Friedan and E. J. Martinec, *Nucl. Phys.* B299 (1988) 601; T. Banks and L. Dixon, *Nucl. Phys.* B307 (1988) 93.

8. T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal Algebras and String Compactification on Manifolds with* $SU(n)$ *Holonomy*, University of Tokyo preprint UT-536, August 1988, to be published in Nucl. Phys. B.

9. L. Alvarez-Gaumé, S. Coleman and P. Ginsparg, *Commun. Math. Phys.* 103 (1986) 423.

10. L. Alvarez-Gaumé, D. Z. Freedman, *Commun. Math. Phys.* 80 (1981) 443; L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, *Ann. Phys.* 134 (1981) 85.

11. L. Alvarez-Gaumé and P. Ginsparg, *Commun. Math. Phys.* 102 (1985) 311; C. Hull, *Nucl. Phys.* B260 (1985) 182.

12. M. T. Grisaru, A. E. M. van de Ven and D. Zanon, *Phys. lett.* 178B (191986) 423.

13. D. Nemeschansky and A. Sen, *Phys. Lett.* 178B (1986) 365; A. Sen, *Phys. Lett.* 178B (1986) 370.

14. E. Witten, *Nucl. Phys.* B268 (1986) 79.

15. T. Banks and N. Seiberg, *Nucl. Phys.* B273 (1986) 157.

16. A. Schwimmer and N. Seiberg, *Phys. Lett.* 184B (1987) 191.

17. E. Witten, *Nucl. Phys.* N202 (1982) 253.

18. T. Eguchi and A. Taormina, *Phys. Lett.* 198B (1987) 75, 200B (1988) 315, 210B (1988) 125.

19. See for example M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory 2*, Cambridge University Press (1987).

20. A. Schellekens and N. Warner, *Phys. Lett.* 177B (1986) 317;, 181B (1986) 339; O. Alvarez, T, Killingback, M. Mangano and P. Windey, Berkeley preprint LBL-23044 (1987).

21. E. Witten, *Commun. Math. Phys.* 109 (1987) 525.

22. D. Zagier, *Note on the Landweber-Stong Elliptic Genus*, Bonn preprint (1987); C. Taubes, S^1 *Action and Elliptic Genera*, Harvard University preprint (1988).

23. L. J. Mordell, *Acta Mathematica* 61 (191933) 323.

24. L. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* B261 (1985) 678;, B274 (1986) 285.

25. M. Walton, *Phys. Rev. D* 37 (1988) 377.

26. D. Gepner, *Nucl. Phys.* B296 (1988) 757, *Phys. Lett.* 199B (1987) 380.

27. P. Di-Vecchia, J. L. Petersen and H. B. Zheng, *Phys. Lett.* 174B (1986) 280; S. Nam, *Phys. lett.* 172B (1986) 323.

28. F. Ravanini and S.-K. Yang, *Phys. Lett.* 195B (1987) 202; Z. Qiu, *Phys. Lett.* 198B (1987) 497.

29. A. Cappelli, C. Itzykson and J.-B. Zuber, *Nucl. Phys.* B280 (1987) 445, *Commun. Math. Phys.* 113 (1987) 1; D. Gepner, *Nucl. Phys.* B287 (1987) 111.

30. E. J. Martinec, *Algebraic Geometry and Effective Lagrangians*, University of Chicago preprint EFI 88-76, November 1988.

31. C. Vafa and N. Warner, *Catastrophes and the Classification of Conformal Theories*, Harvard University preprint HUTP-88/A037, 1988.

32. S. Odake, *Extension of $N = 2$ Superconformal Algebra and Calabi-Yau Compactification*, University of Tokyo preprint UT-537, September 1988.