Problem Set, Fall 2016

Assigned on December 1, 2016.   Due on December 8, 2016.

This is an open-book take-home exam, but you should not search soluitons on internet. You can take as long as you like to solve these problems. You can collaborate with other students, but you should write your own solutions.

The two bonus problems are optional, but you could get extra points by solving these if you are asking for letter grades.

The solutions should be placed in the TA mailbox on the fourth floor of Lauritsen or emailed at `pkravchuk@caltech.edu`. The TA for this class is Petr Kravchuk.

[1] Let ω be a k-form on a manifold M. At each point $p \in M$, it gives a totally anti-symmetric multi-linear function on $\oplus^k T_pM$,

$$(v_1, ..., v_k) \in \oplus^k T_pM \to \omega(v_1, ..., v_k) \in \mathbf{R}.$$

For a given tangent vector field u, define a map $i(u)$ from k-forms to $(k-1)$-forms by,

$$\big[i(u)\omega\big](v_1, ..., v_{k-1}) = \omega(u, v_1, ..., v_{k-1}).$$

[1-1] Show $i(u_1)i(u_2) = -i(u_2)i(u_1)$.

[1-2] When α is a k-form and β is an l-form, show

$$i(v)(\alpha \wedge \beta) = (i(v)\alpha) \wedge \beta + (-1)^k \alpha \wedge (i(v)\beta).$$

[2] When ω is a k-form, show $**\,\omega$ is proportional to ω for the Hodge star operator $*$. Compute the proportionality constant.

[3] Compute the first cohomology H^1 of a circle S^1.

[4] The affine spin connection $\omega^a{}_b$ is a 1-form defined by

$$de^a + \omega^a{}_b \wedge e^b = 0, \qquad (1)$$

where

$$e^a = e^a_i dx^i, \qquad g_{ij} = \sum_a e^a_i e^a_j$$

The curvature 2-form is defined by

$$R^c{}_d = \frac{1}{2} R_{ab}{}^c{}_d e^a \wedge e^b = d\omega^c{}_d + \omega^c{}_f \wedge \omega^f{}_d. \qquad (2)$$

[4-1] Show that the curvature defined in this way is related to the Riemann curvature by

$$R_{ij}{}^k{}_l = R_{ab}{}^c{}_d e^a_i e^b_j e^k_c e^d_l.$$

[4-2] By taking the exterior derivative of (1), show

$$R^a{}_b \wedge e^b = 0.$$

Show that it is equivalent to $R_{[ijk]l} = 0$.

[4-3] By taking the exterior derivative of (2) and show,

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = DR^a{}_b = 0.$$

Check that it is equivalent to the Bianchi identity for the Riemann curvature.

[4-4] Einstein's first attempt to write the gravity equation was,

$$R_{ij} = \kappa T_{ij},$$

where T_{ij} is the energy-momentum tensor satisfying the conservation law, $\nabla^i T_{ij} = 0$, and κ is some constant. Explain what is wrong with this equation.

[5] Suppose M is a Kähler manifold.

[5-1] Only non-zero components of the affine connection on M are,

$$\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}},$$

and its complex conjugate. Components with mixed indices (mixed in i and $\bar{j}$) all vanish.

[5-2] Show that the Ricci tensor of M can be expressed as,

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \mathrm{logdet}(g).$$

[Bonus Problem 1] Define the Lie derivative $\mathcal{L}_u$ with respect to a vector field u of a p-form w by the Leibniz identity,

$$u(w(v_1, \cdots, v_n)) = (\mathcal{L}_u w)(v_1, \cdots, v_n) + w([u, v_1], v_2, \cdots, v_n) + \cdots + w(v_1, \cdots, [u, v_n]).$$

[B1-1] Show that $(\mathcal{L}_u w)$ as defined above gives a p-form, *i.e.*, its value does not depend on derivatives of its arguments

[B1-2] Using the above definition, prove the Cartan identity,

$$\mathcal{L}_u = i(u)d + di(u).$$

[B1-3] Suppose that a differential p-form w on $\mathbb{R}^n$ is homogeneous of degree h, *i.e.*, for $x \in \mathbb{R}^n$, the p-form w at ax for $a \in \mathbb{R}$ is equal to $a^h w$ at x. Let u be the vector field given by identity map on $\mathbb{R}^n$. Compute $\mathcal{L}_u w$ for such w.

[B1-4] With w the same as above, find the value of h which ensures that the condition $i(u)w = 0$ is preserved by the exterior derivative d. Note that such forms (satisfying $i(u)w = 0$) are in one-to-one correspondence with differential p-forms in the projective space $\mathrm{P}\mathbb{R}^{n-1} = [\mathbb{R}^n - (0, \cdots, 0)]/\mathbb{R}$. Linear transformations in $\mathbb{R}^n$ induce projective transformations on $\mathrm{P}\mathbb{R}^{n-1}$, which are in general non-linear in local coordinates. This exercise gives a linear realization for the projective transformations acting on the de Rham complex in $\mathrm{P}\mathbb{R}^n$.

[Bonus Problem 2] Consider a 4-dimensional metric of the form,

$$ds^2 = V(x) d\vec{x} \cdot d\vec{x} + \frac{1}{V(x)} (d\tau - \vec{A}(x) \cdot d\vec{x})^2,$$

where $\vec{x} \in \mathbf{R}^3$, and X and $\vec{A}$ are scalar and vector fields on $\mathbf{R}^3$. Show that the metric obeys the vacuum Einstein equation,

$$R_{\mu\nu} = 0,$$

if V and $\vec{A}$ satisfy,

$$\Delta V = 0, \quad \vec{\nabla} V = \vec{\nabla} \times \vec{A},$$

where Δ is the Laplacian on $\mathbf{R}^3$, $\vec{\nabla} V$ is the gradient of V, and $\vec{\nabla} \times \vec{A}$ is the rotation of $\vec{A}$.